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# Complete blocked triangular matrix rings over a Noetherian ring

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**Abstract**

We show that the underlying Boolean matrix $B$ of a complete blocked triangular matrix ring $\mathbb{M}(B,R)$ over a left Noetherian ring $R$ is unique, i.e. if $\mathbb{M}(B_1,R)$ and $\mathbb{M}(B_2,R)$ are isomorphic complete blocked triangular matrix rings over a left Noetherian ring $R$, then $B_1 = B_2$. 

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Although it is not possible to recover up to isomorphism the base ring $R$ from the complete matrix ring $\mathbb{M}_n(R)$, not even in the prime Noetherian case (see [1]), it was shown in [2] that the underlying Boolean matrix $B$ of a structural matrix ring $\mathbb{M}(B,R)$ over a semiprime Noetherian ring $R$ can be recovered. To be more precise, [2, Theorem 2.4] states that the underlying Boolean matrices $B_1$ and $B_2$ of two isomorphic structural matrix rings $\mathbb{M}(B_1,R)$ and $\mathbb{M}(B_2,R)$ over a semiprime left Noetherian ring $R$ are conjugated, i.e. one of them can be obtained from the other by a permutation of the rows and columns, which is equivalent to saying that the directed graphs associated with $B_1$ and $B_2$ are isomorphic.

The question has remained open whether semiprimeness can be dropped in [2, Theorem 2.4], and it was answered in the affirmative in the commutative case in [2, Corollary 2.5]. In this note, we show that the answer is also positive in the complete blocked triangular case. Moreover, we show that in this case the underlying Boolean matrices are equal, i.e. the underlying Boolean matrix of a complete blocked triangular matrix ring over a left Noetherian ring is unique. Complete blocked triangular

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matrix rings over division rings feature in the representation of left Artinian CI-prime rings in [3].

Throughout the paper we use the notation of [2], and for the ease of the reader we provide the relevant details. Every ring herein is associative with identity, and we denote the prime radical of a ring $R$ and the uniform dimension of a left $R$-module $M$ by $\mathscr{P}(R)$ and $u\dim_R M$, respectively. See, for example, [4]. An $n \times n$ Boolean matrix $B=[b_{i,j}]$ is called *complete blocked triangular* if it is of the form

$$\begin{bmatrix} X_{1,1} & X_{1,2} & \dots & X_{1,t} \\ 0 & X_{2,2} & \dots & X_{2,t} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & X_{t,t} \end{bmatrix},$$

where for every $i \le j$, $X_{i,j}$ is an $n_i \times n_j$ (Boolean) matrix with all its entries equal to 1, and $n_1+\cdots+n_t=n$. We call $X_{i,j}$ the $(i,j)$th block and $X_{i,i}$ the $i$th diagonal block of $B$. The complete blocked triangular matrix ring (associated with $B$ and $R$) is the subring $\mathbb{M}(B,R)$ of $\mathbb{M}_n(R)$ comprising all matrices having 0 in the position $(k,l)$ whenever $b_{k,l}=0$.

**Theorem.** *Let $B_1$ and $B_2$ be complete blocked triangular Boolean matrices, and let $R$ be a left Noetherian ring. If $\mathbb{M}(B_1,R)\cong\mathbb{M}(B_2,R)$, then $B_1=B_2$.*

**Proof.** From [2, Lemma 2.1] we know that the number of diagonal blocks of $B_1$ equals that of $B_2$, and if $n_1,\dots,n_t$ and $m_1,\dots,m_t$ are the consecutive orders of the diagonal blocks of $B_1$ and $B_2$, respectively, then the $m_i$'s are a permutation of the $n_j$'s. This implies that the orders of $B_1$ and $B_2$ are equal. We shall show $n_t=m_t$, and that if $n_j=m_j$ for $j=t,\, t-1,\dots,t-k$, then $n_{t-k-1}=m_{t-k-1}$, and so the desired result follows by induction.

By [6, Theorem 2.7] $\mathscr{I}_i := \mathscr{P}(\mathbb{M}(B_i,R))$ comprises all the matrices in $\mathbb{M}(B_i,R)$ having elements of $I := \mathscr{P}(R)$ in the diagonal blocks of $B_i$, $i=1,2$. Therefore, for every $l\ge 1$, $\mathscr{I}_i^l$ comprises all the matrices in $\mathbb{M}(B_i,R)$ having elements of $I^l$ in the diagonal blocks of $B_i$ and having elements of $I^{l-q}$ in the $(p,p+q)$th block of $B_i$, $q=1,\dots,t-1$, $p=1,\dots,t-q$, where we set $I^{l'}:=R$ if $l'\le 0$. Since $R$ is left Noetherian, we have that $I$ is nilpotent. Let $s$ be the nilpotency index of $I$. We conclude that $\mathscr{I}_i^{t+s-2}$ is zero in all the blocks of $B_i$, except in the $(1,t)$th block where it has entries from $I^{s-1}$. Since $\mathbb{M}(B_1,R)\cong\mathbb{M}(B_2,R)$, it follows that

$$n_t\cdot u\dim_R I^{s-1} = u\dim_{\mathbb{M}(B_1,R)}\mathscr{I}_1^{t+s-2} = u\dim_{\mathbb{M}(B_2,R)}\mathscr{I}_2^{t+s-2} = m_t\cdot u\dim_R I^{s-1}.$$

Therefore, $n_t=m_t$.

Suppose now that $n_j=m_j$ for $j=t,\, t-1,\dots,t-k$, for some $k\ge 0$. Then, by the previous paragraph $\mathscr{I}_i^{(t-k-1)+(s-2)}$ is zero in all the blocks of $B_i$, except for having elements of $I^{(t-k-1)+(s-2)-q}$ in the $(p,p+q)$th block of $B_i$, $q=t-k-2,\dots,t-1$, $p=1,\dots,t-q$. Let $n_1+\cdots+n_{t-k-2}+1\le v\le n_1+\cdots+n_t$, and let $\mathscr{C}_{1,v}$ denote the left

$\mathbb{M}(B_1,R)$-submodule of $\mathscr{I}_1^{(t-k-1)+(s-2)}$ consisting of all the matrices (in $\mathscr{I}_1^{(t-k-1)+(s-2)}$) which have zeroes everywhere except possibly in the $v$th column. If $n_1+\cdots+n_{t-k-2}+1 \le v \le n_1+\cdots+n_{t-k-1}$, then

$$u\dim_{\mathbb{M}(B_1,R)}\mathscr{C}_{1,u} = u\dim_R I^{s-1}, \tag{1}$$

since in this case

$$\mathscr{C}_{1,v} = I^{s-1}E_{1,v} + \cdots + I^{s-1}E_{n_1,v},$$

where the $E_{i,j}$'s denote the standard matrix units (with 1 in position $(i,j)$ and zeros elsewhere). Next, if $n_1+\cdots+n_{t'}+1 \le v \le n_1+\cdots+n_{t'+1}$ for some $t'$ such that $t-k-1 \le t' \le t-1$, then

$$\begin{aligned}\mathscr{C}_{1,v} = {} & (I^{(t-k-1)+(s-2)-t'}E_{1,v} + \cdots + I^{(t-k-1)+(s-2)-t'}E_{n_1,v}) \\ & + (I^{(t-k-1)+(s-2)-t'+1}E_{n_1+1,v} + \cdots + I^{(t-k-1)+(s-2)-t'+1}E_{n_1+n_2,v}) + \cdots \\ & + (I^{s-1}E_{n_1+\cdots+n_{(t'+1)-(t-k-1)}+1,v} + \cdots + I^{s-1}E_{n_1+\cdots+n_{(t'+2)-(t-k-1)},v}).\end{aligned}$$

(We note that $s-1-((t-k-1)+(s-2)-t') = (t'+2)-(t-k-1)-1$.)

Furthermore, every $\mathbb{M}(B_1,R)$-submodule of $\mathscr{C}_{1,v}$ is of the form

$$\begin{aligned}& (M_0E_{1,v} + \cdots + M_0E_{n_1,v}) + (M_1E_{n_1+1,v} + \cdots + M_1E_{n_1+n_2,v}) + \cdots \\ & \quad + (M_{(t'+1)-(t-k-1)}E_{n_1+\cdots+n_{(t'+1)-(t-k-1)}+1,v} + \cdots \\ & \quad\quad + M_{(t'+1)-(t-k-1)}E_{n_1+\cdots+n_{(t'+2)-(t-k-1)},v})\end{aligned}$$

for some $(t'+1)-(t-k-1)+1$ left ideals $M_0, M_1, \ldots, M_{(t'+1)-(t-k-1)}$ of $R$ such that

$$M_w \subseteq I^{(t-k-1)+(s-2)-t'+w} \tag{2}$$

for $w = 0, 1, \ldots, (t'+1)-(t-k-1)$, and

$$M_w \subseteq M_{w'} \tag{3}$$

if $w \ge w'$. Therefore,

$$u\dim_{\mathbb{M}(B_1,R)}\mathscr{C}_{1,v} = u\dim \mathscr{L}_{(t'+1)-(t-k-1)+1}, \tag{4}$$

where $\mathscr{L}_{(t'+1)-(t-k-1)+1}$ is the modular lattice

$$\{(M_0, M_1, \ldots, M_{(t'+1)-(t-k-1)}) \mid M_w \text{ is a left ideal of } R \text{ for } w = 0, 1, \ldots, (t'-1)+(t-k-1), \text{ and (2) and (3) hold}\}.$$

(The uniform dimension of a modular lattice is also called the Goldie dimension. See, for example, [5].)

Consequently, by (1) and (4), recalling that $t-k-1 \le t' \le t-1$, we have that

$$u\dim_{\mathbb{M}(B_1,R)} \mathscr{I}_1^{(t-k-1)+(s-2)} = n_{t-k-1} \cdot u\dim_R I^{s-1} + n_{t-k} \cdot u\dim \mathscr{L}_2 + \cdots + n_t \cdot u\dim \mathscr{L}_{k+2}. \tag{5}$$

Applying the same arguments to $\mathbb{M}(B_2,R)$, we obtain that

$$u\dim_{\mathbb{M}(B_2,R)} \mathscr{I}_2^{(t-k-1)+(s-2)} = m_{t-k-1} \cdot u\dim_R I^{s-1} + m_{t-k} \cdot u\dim \mathscr{L}_2 + \cdots + m_t \cdot u\dim \mathscr{L}_{k+2}. \tag{6}$$

Since $n_t = m_t, \ldots, n_{t-k} = m_{t-k}$, and since the left-hand sides of (5) and (6) are equal, it follows that $n_{t-k-1} = m_{t-k-1}$. □

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## References

[1] A.W. Chatters, Non-isomorphic rings with isomorphic matrix rings, Proc. Edinburgh Math. Soc. 36 (1993) 339–348.
[2] S. Dăscălescu, L. van Wyk, Do isomorphic structural matrix rings have isomorphic graphs?, Proc. Amer. Math. Soc. 124 (1996) 1385–1391.
[3] M.S. Li, J.M. Zelmanowitz, Artinian rings with restricted primeness conditions, J. Algebra 124 (1989) 139–148.
[4] J.C. McConnell, J.C. Robson, Noncommutative Noetherian Rings, Wiley, New York, 1987.
[5] C. Năstăsescu, F. Van Oystaeyen, Dimensions of Ring Theory, Reidel, Dordrecht, Holland, 1987.
[6] L. van Wyk, Special radicals in structural matrix rings, Comm. Algebra 16 (1988) 421–435.